Balancing an Inverted Pendulum

Modeling, Stabilization, and Swing-Up of the Quanser Qube Servo 3

ME 3140 — Modeling and Control of Dynamic Systems

Fall 2026 · Final Project

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Project Overview

1 Step 1
Modeling

Lagrangian Mechanics, System ID, validate vs p-code

2 Step 2
Controller Design

Pole placement state feedback to stabilize upright

3 Step 3
P-Code Testing

Evaluate controller on real plant, retune gains

4 Step 4
Cascade + Swing-Up

Arm position tracking + swing up from hanging down

Goal: balance the pendulum upright ($\theta = \pi$) while regulating the arm position — and, as a bonus, start from hanging down and swing up.

STEP 1

System Modeling — Degrees of Freedom

Two degrees of freedom

ϕ (arm angle): rotation of the horizontal arm, driven by the DC motor.

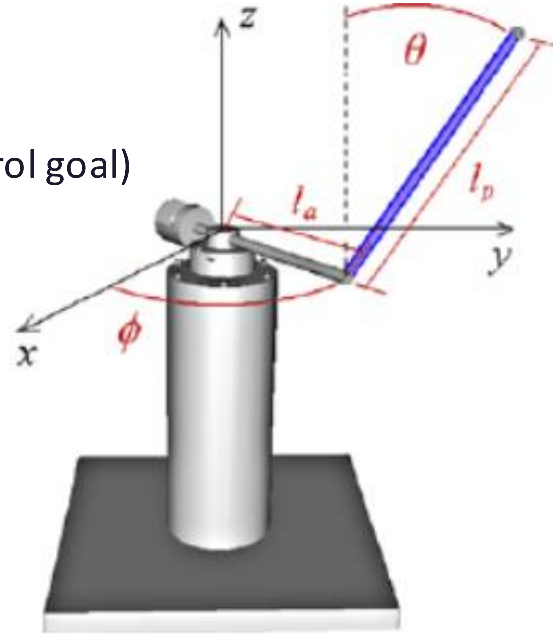
θ (pendulum angle): pendulum rotation about the end of the arm.

Convention

- $\theta = 0$: hanging DOWN (stable)
- $\theta = \pi$: UPRIGHT (unstable — control goal)

Fourth-order system

State vector: $x = [\phi, \dot{\phi}, \theta, \dot{\theta}]^T$



ACTUATOR

DC motor

Voltage input, ±10V saturation

SENSORS

Encoders

2048 counts/rev (0.176° resolution)

CHALLENGE

Unstable

Open-loop pole at +16 rad/s at upright

STEP 1

Equations of Motion — Lagrangian Mechanics

Approach

1. Kinetic + potential energy of arm and pendulum
2. Lagrangian $L = T - V$
3. Apply Euler-Lagrange equations for each DOF
4. Add motor torque, viscous damping
5. Linearize about upright for controller design

Matrix form

$$M(\theta) \cdot [\ddot{\phi} \,;\, \ddot{\theta}] = F(\dot{\phi}, \theta, \dot{\theta}, V)$$

M is state-dependent (couples arm and pendulum via $Mlr \cdot \cos\vartheta$)

Linearized about upright ($\theta = \pi$)

$$\dot{x} = A \cdot x + B \cdot V$$

$x = [\phi, \dot{\phi}, \theta, \dot{\theta}]^T$

A has:

- Arm integrator pole ($s = 0$)
- Fast stable poles at $s = -20.4, -1.8$
- **UNSTABLE pole at $s = +15.97$**

The +15.97 rad/s pole is inherent to inverted pendulum physics — matches theoretical $\sqrt{\alpha \cdot m_0 / \det M} \approx 18$.

STEP 1

Equations of Motion

Nonlinear EOMs

$$J_1 + J_2 \sin^2\theta)\ddot{\phi} + \frac{1}{2} m_p L_r L_p \cos\theta\, \ddot{\theta} + 2J_2 \sin\theta \cos\theta\, \dot{\theta}\dot{\phi} - \frac{1}{2} m_p L_r L_p \sin\theta\, \dot{\theta}^2 + B_\phi \dot{\phi} = \frac{k_t}{R_m} V$$

$$\frac{1}{2} m_p L_r L_p \cos\theta\, \ddot{\phi} + J_2 \ddot{\theta} - J_2 \sin\theta \cos\theta\, \dot{\phi}^2 + \frac{1}{2} m_p g L_p \sin\theta + b_\theta \dot{\theta} = 0$$

Linearized EOMs

$$J_1 \ddot{\phi} - \frac{1}{2} m_p L_r L_p \ddot{\tilde{\theta}} + B_\phi \dot{\phi} = \frac{k_t}{R_m} V$$

$$-\frac{1}{2} m_p L_r L_p \ddot{\phi} + J_2 \ddot{\tilde{\theta}} + b_\theta \dot{\tilde{\theta}} - \frac{1}{2} m_p g L_p \tilde{\theta} = 0$$

$$J_1 = \frac{1}{3} m_r L_r^2 + J_m + J_h + m_p L_r^2$$

$$J_2 = \frac{1}{3} m_p L_p^2$$

$$B_\phi = \frac{k_t k_m}{R_m} + b_\phi$$

STEP 1

Transfer Functions

$$\frac{\tilde{\Theta}[s]}{V[s]} = \frac{\frac{k_t}{R_m}\left(\frac{1}{2}m_p L_r L_p\right)s}{\left(J_1 J_2 - \left(\frac{1}{2}m_p L_r L_p\right)^2\right)s^3 + (J_1 b_\theta + J_2 B_\phi)s^2 + \left(B_\phi b_\theta - J_1 \frac{1}{2}m_p g L_p\right)s - B_\phi \frac{1}{2}m_p g L_p}$$

$$\frac{\Phi[s]}{V[s]} = \frac{\frac{k_t}{R_m}\left(J_2 s^2 + b_\theta s - \frac{1}{2}m_p g L_p\right)}{s\left[\left(J_1 J_2 - \left(\frac{1}{2}m_p L_r L_p\right)^2\right)s^3 + (J_1 b_\theta + J_2 B_\phi)s^2 + \left(B_\phi b_\theta - J_1 \frac{1}{2}m_p g L_p\right)s - B_\phi \frac{1}{2}m_p g L_p\right]}$$

$$J_1 = \frac{1}{3}m_r L_r^2 + J_m + J_h + m_p L_r^2$$

$$J_2 = \frac{1}{3}m_p L_p^2$$

$$B_\phi = \frac{k_t k_m}{R_m} + b_\phi$$

STEP 1

Parameter Identification

Known (datasheet & geometry)

Parameter	Value	Source
Motor resistance Rm	7.5 Ω	datasheet
Torque/back-EMF kt = km	0.0422	datasheet
Arm length Lr	0.085 m	datasheet
Arm mass mr	0.095 kg	datasheet
Pendulum Lp	0.129 m	datasheet
Pendulum mp	0.024 kg	datasheet
$\beta = \frac{1}{3} m_p \cdot L_p^2$	1.33×10^{-4}	geometry
$M_{lr} = m_p \cdot L_r \cdot L_p/2$	1.32×10^{-4}	geometry

Empirically tuned (vs p-code)

α_mult = 0.50

Textbook α overestimated arm inertia ~2×.

Most arm mass is concentrated near the pivot hub, so effective moment of inertia is half of a uniform-rod approximation.

Viscous damping

$D_r = 1.4 \times 10^{-4}$ (arm)

$D_p = 5.0 \times 10^{-5}$ (pendulum)

Matched by free-oscillation log-decrement and constant-voltage step tests.

Validation vs p-code: RMS ϕ error = 31° and RMS θ error = 7.7° under constant V = 1V step (0–2s)

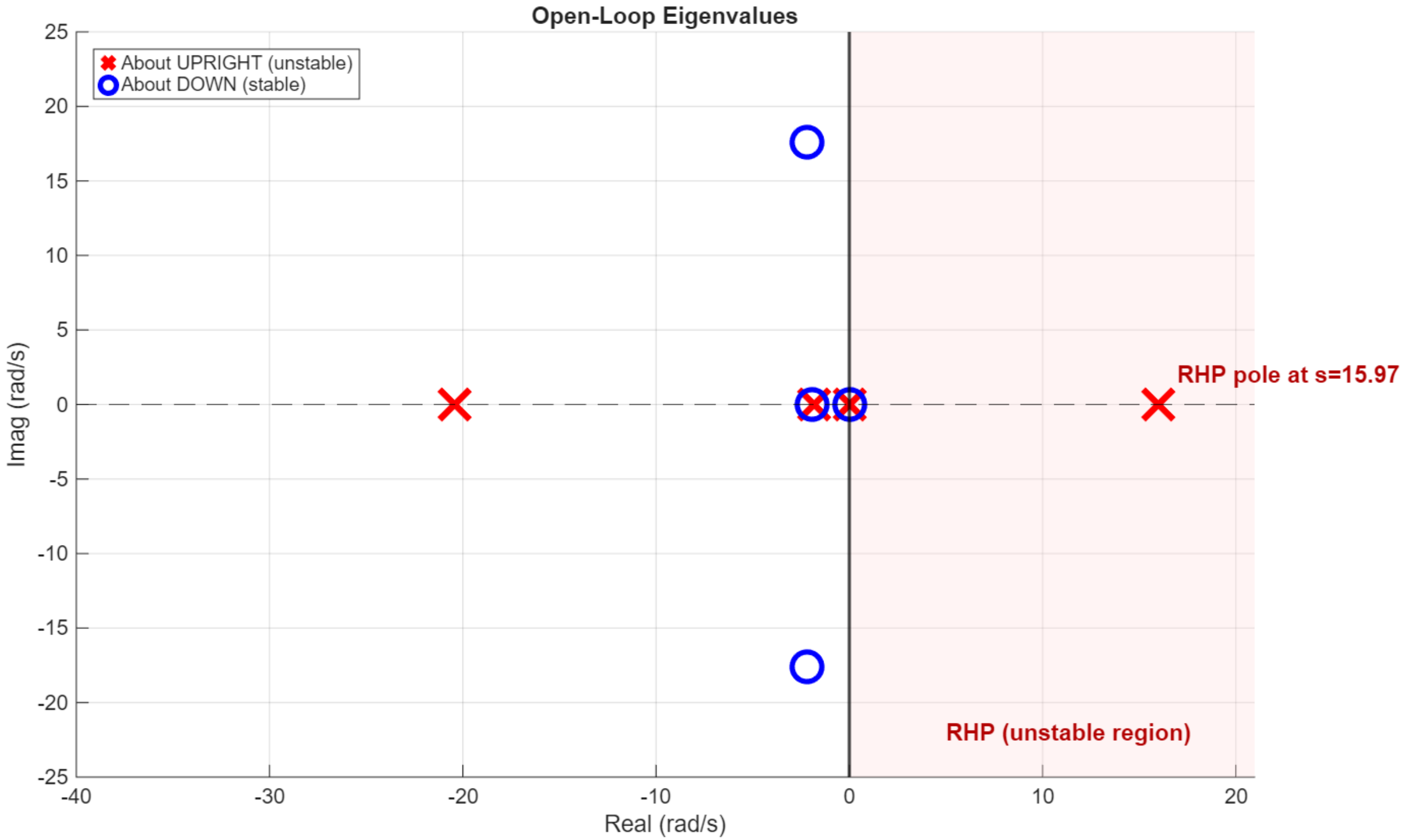
The 7.7° pendulum RMS indicates our model captures the dominant nonlinear behavior (centrifugal equilibrium, coupled oscillation) with only ~5% amplitude error.

Plot of Open Loop Eigenvalues

Open-Loop Eigenvalues

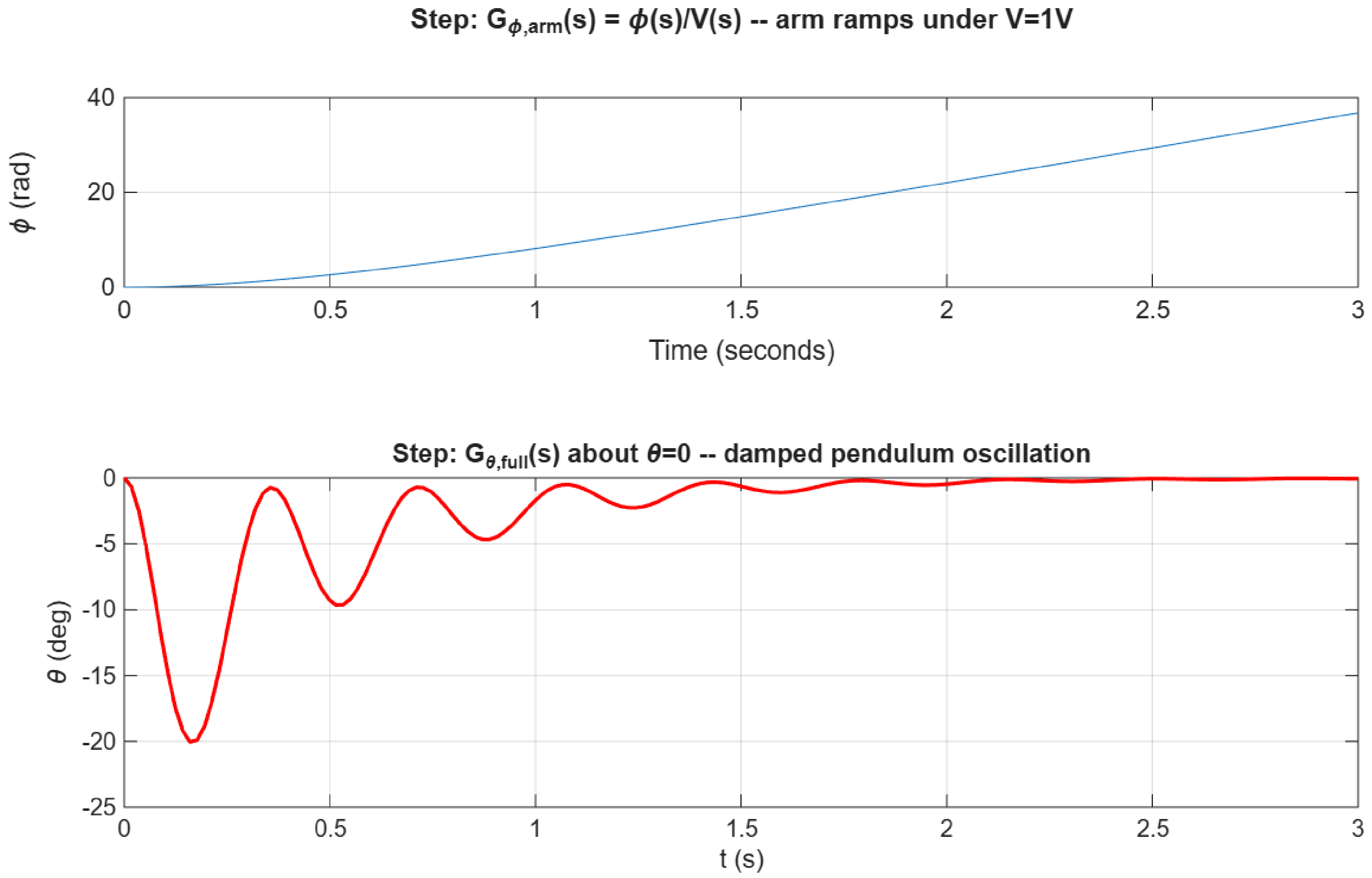
About UPRIGHT (theta = pi)
S = 0
S = -20.4277
S = -1.8321
S = +15.9675

About DOWN (theta = 0)
S = 0
S = -1.9057
S = -2.1934 + 17.572j
S = -2.1934 - 17.572j



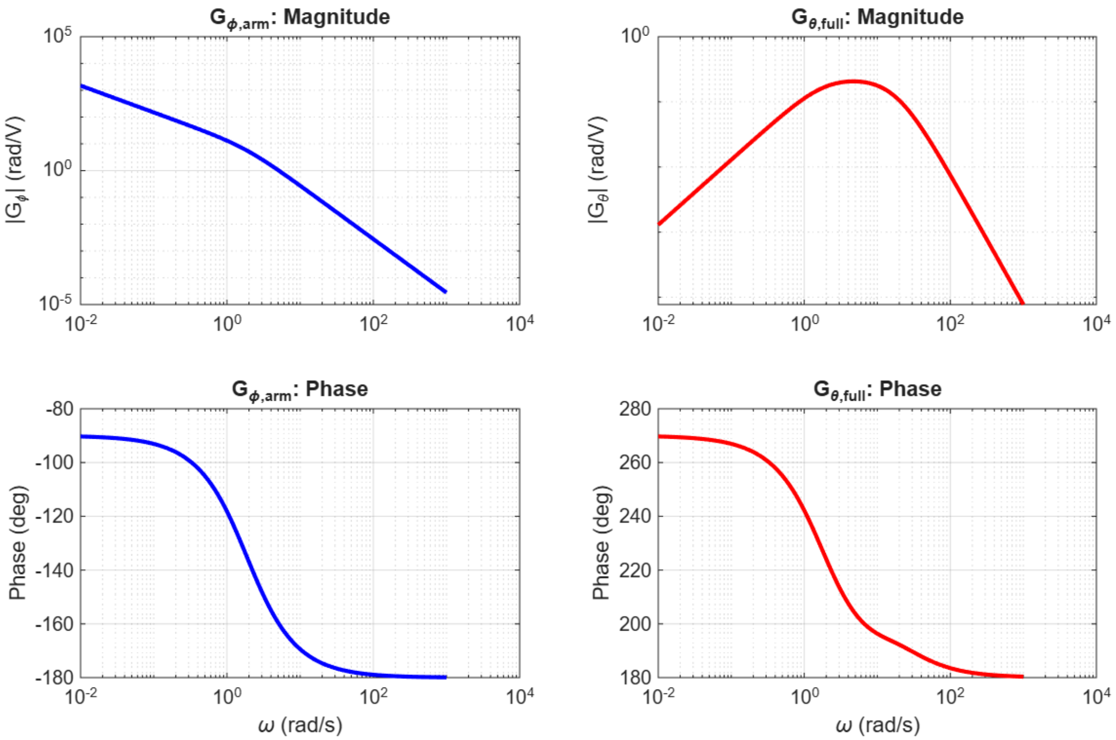
STEP 1

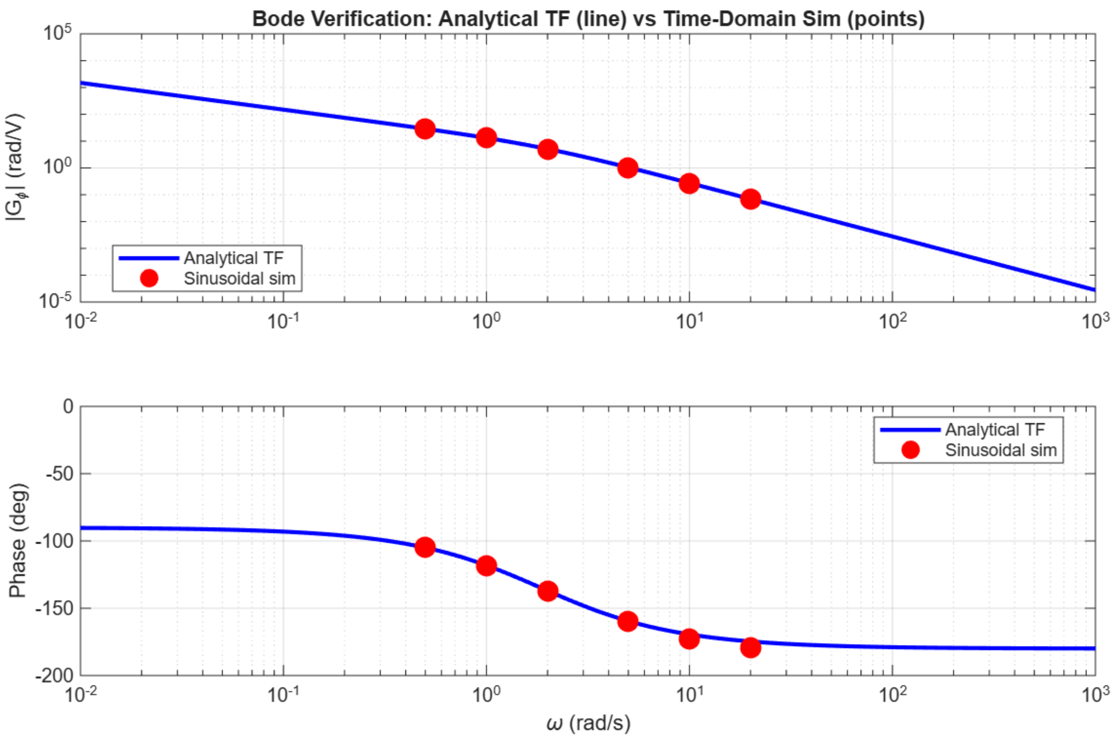
Plot of Step Responses



Nonlinear simulation response

Bode Plot and Bode Verification

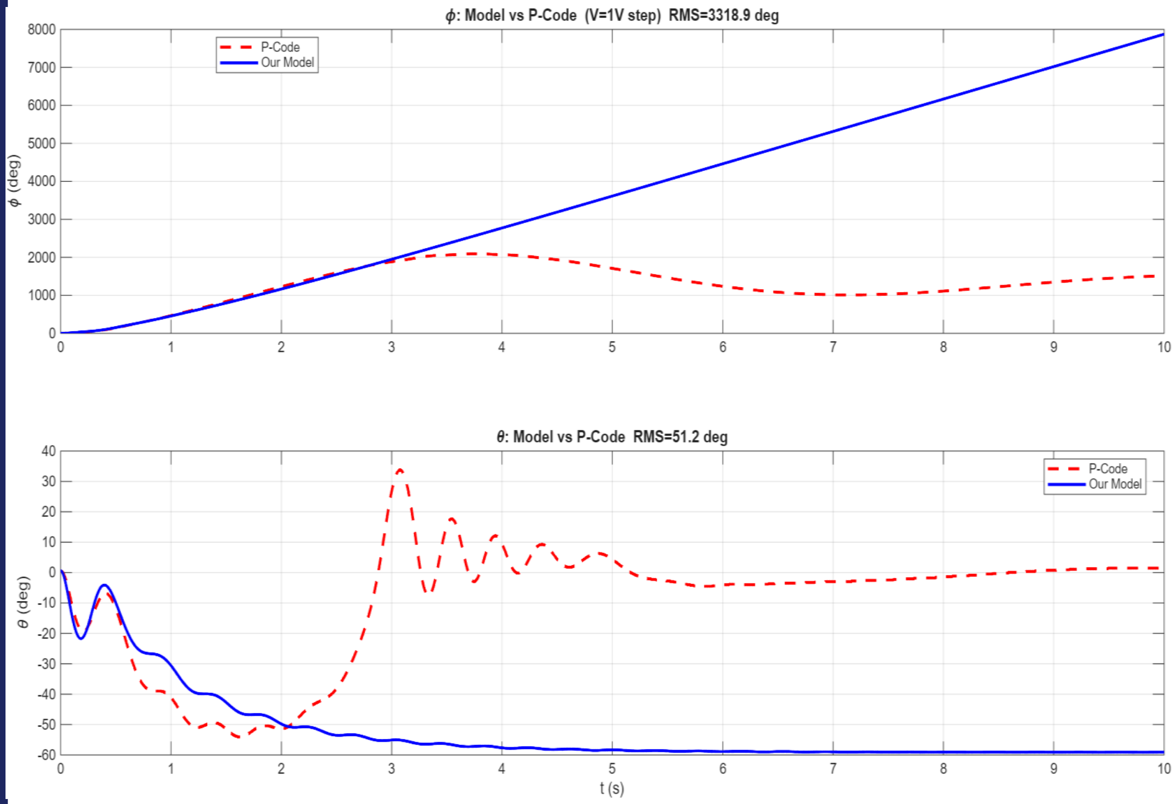


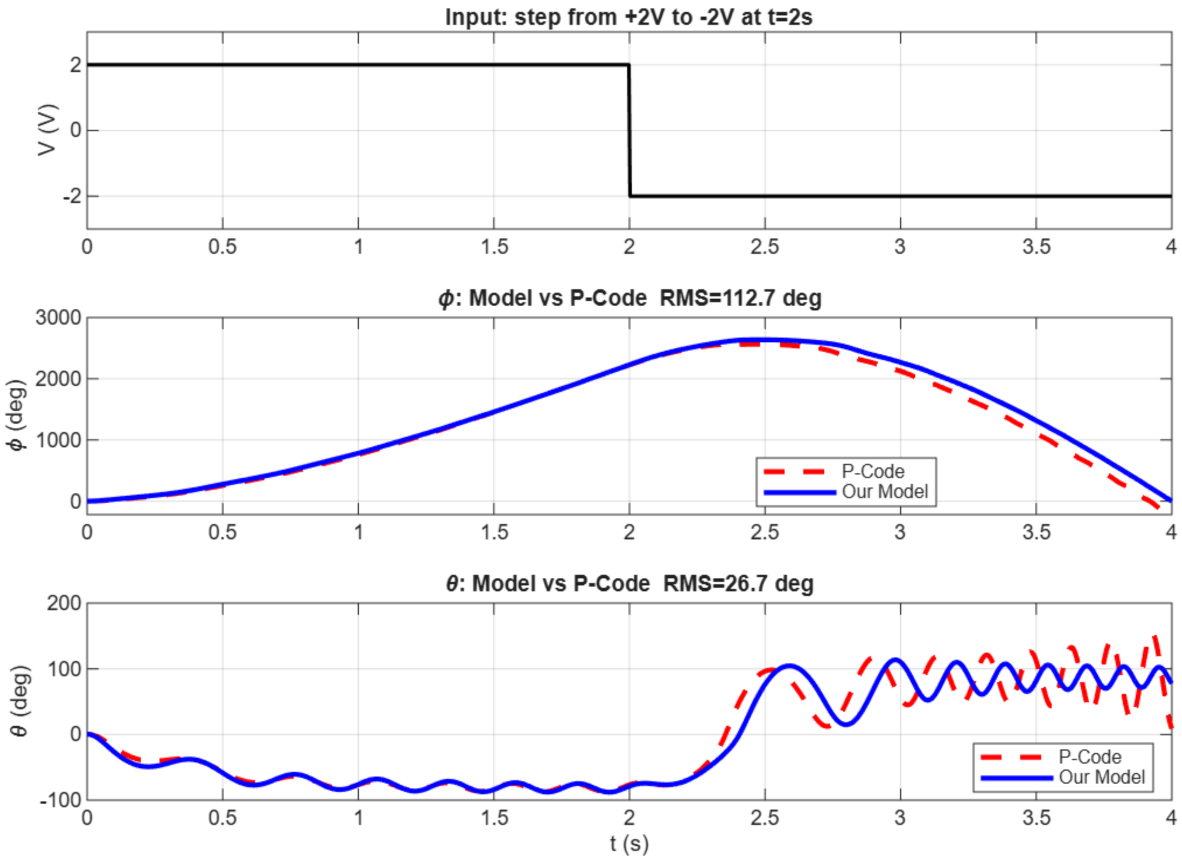


Frequency response confirmed using sinusoidal inputs

STEP 1

Plots Validating Model Against p-code





STEP 2

Why Open-Loop Fails

Open-loop eigenvalues (at upright)

$s = 0$	Arm integrator (free rotation)
$s = -20.4$	Fast stable (damping)
$s = -1.8$	Slow stable
$s = +15.97$	POSITIVE POLE — UNSTABLE

Physical intuition

Errors grow as e^(+15.97t).

A 1° deviation from upright grows to 57° in 0.25s.

Theoretical instability rate:
$\omega\_unstable = \sqrt{(\alpha \cdot m_0 / \det M)} \approx 18$ rad/s

Consistent with the +15.97 pole.

Goal: Design a controller for V(t) that moves all closed-loop poles to the LHP

Specifications: settle time < 0.5s, overshoot < 5%, peak |V| ≤ 10V

STEP 2

Method: State-Feedback Pole Placement

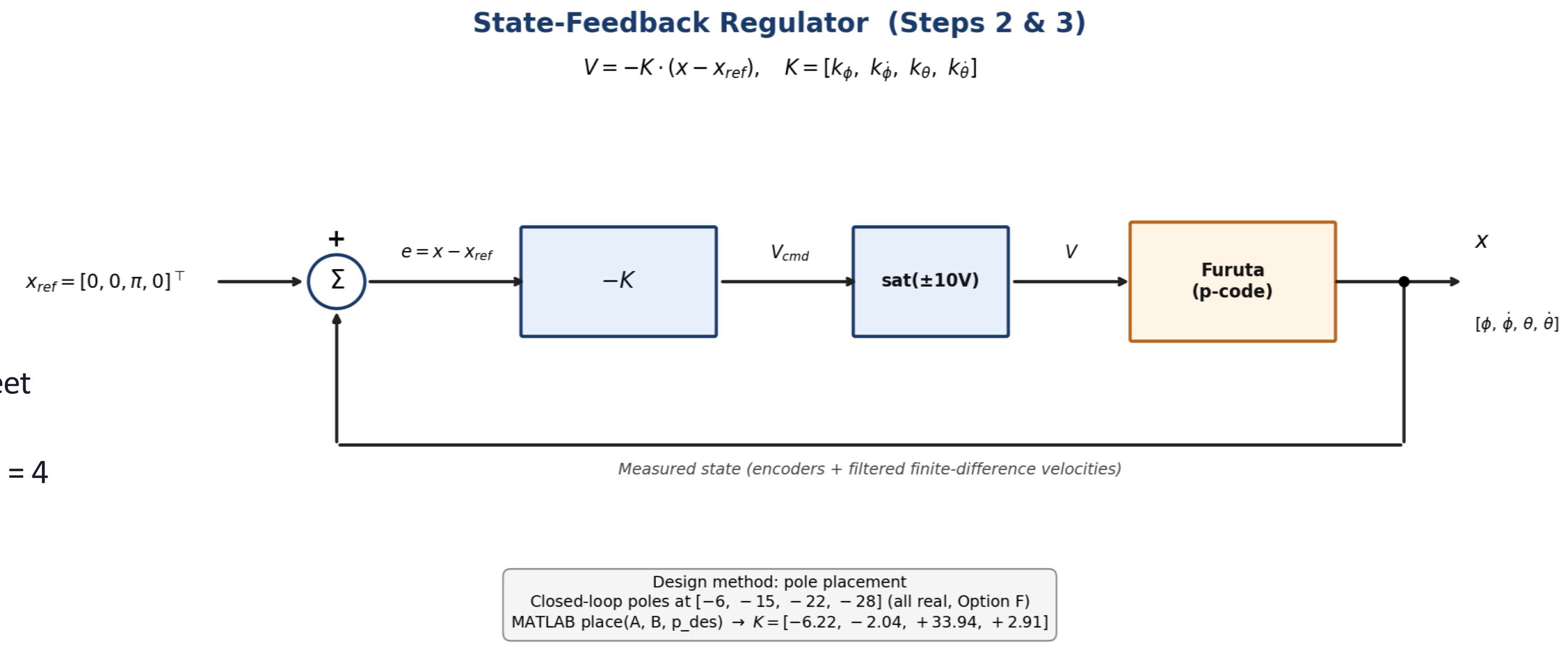
Control law

V = -K · (x - x_ref)

K = [k_ϕ, k_ϕ̇, k_θ, k_θ]

Design steps

1. Pick desired closed-loop poles to meet specs
2. Verify controllability: rank(ctrb(A,B)) = 4
3. Solve for K via place(A, B, p_des)
4. Verify CL eigenvalues match target



STEP 2

Pole Selection for Specs

From the 2nd-order step response formulas

$$t_s \approx 4 / (\zeta \cdot \omega_n)$$

→ *need $\zeta \cdot \omega_n \geq 8$ for $t_s < 0.5$s*

$$OS = \exp(-\pi \cdot \zeta / \sqrt{1-\zeta^2}) \times 100\%$$

→ *need $\zeta \geq 0.707$ for $OS < 5\%$*

Solved closed-loop poles

$$[\ -12 \pm 12j, \quad -24, \quad -30\ ]$$

Dominant pair:

$\omega_n = 16.97$ rad/s, $\zeta = 0.707$

Predicted $t_s = 0.33$s, OS = 4.32%

Fast real poles at −24, −30 don't interfere (separation > 2×).

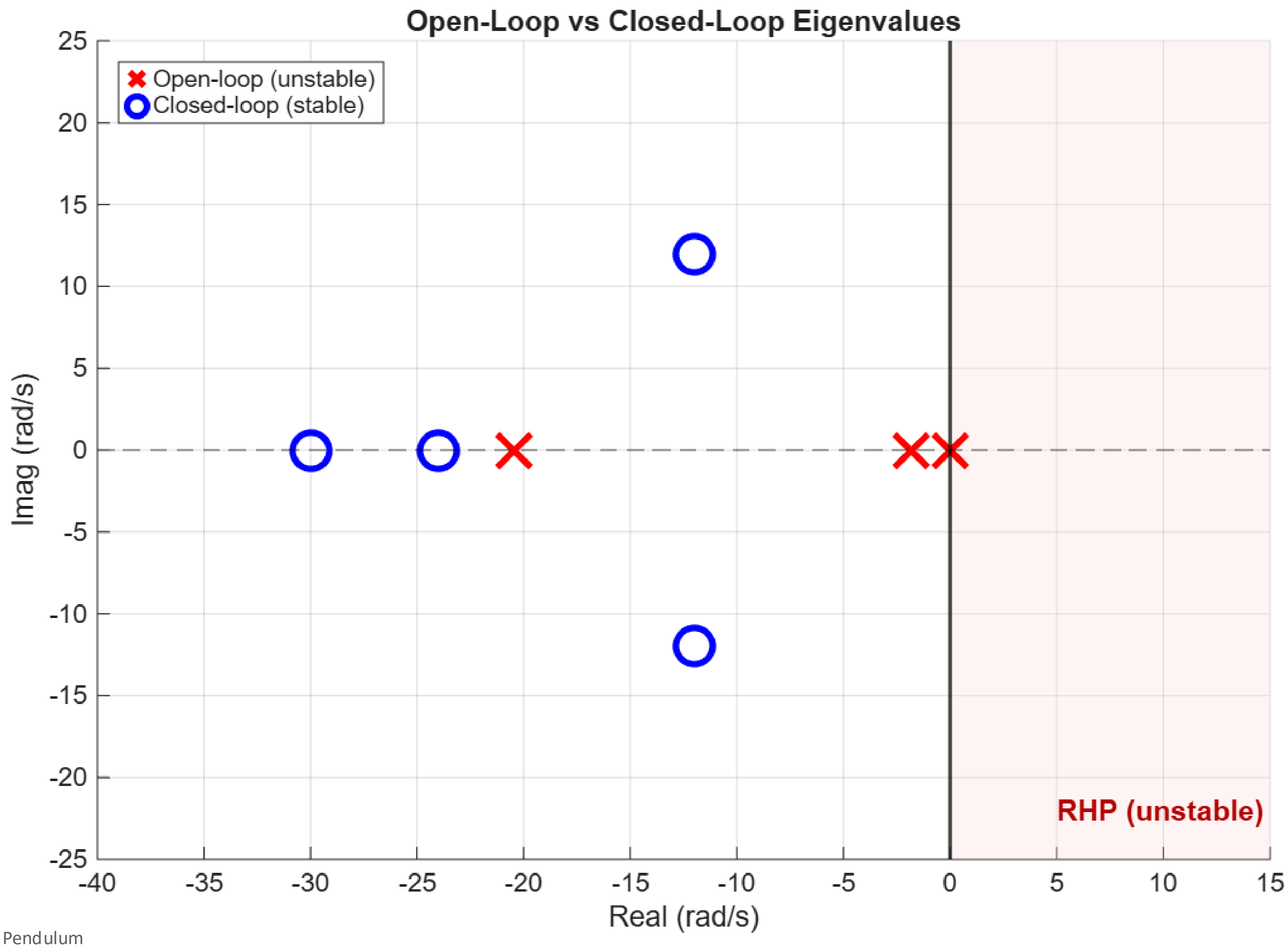
Result: gain matrix K

$$K = [-23.28, \quad -3.83, \quad +58.97, \quad +4.80]$$

$k_\vartheta = +58.97$ V/rad dominates: a 5° pendulum tilt → $V \approx 5.15$ V (under 10V)

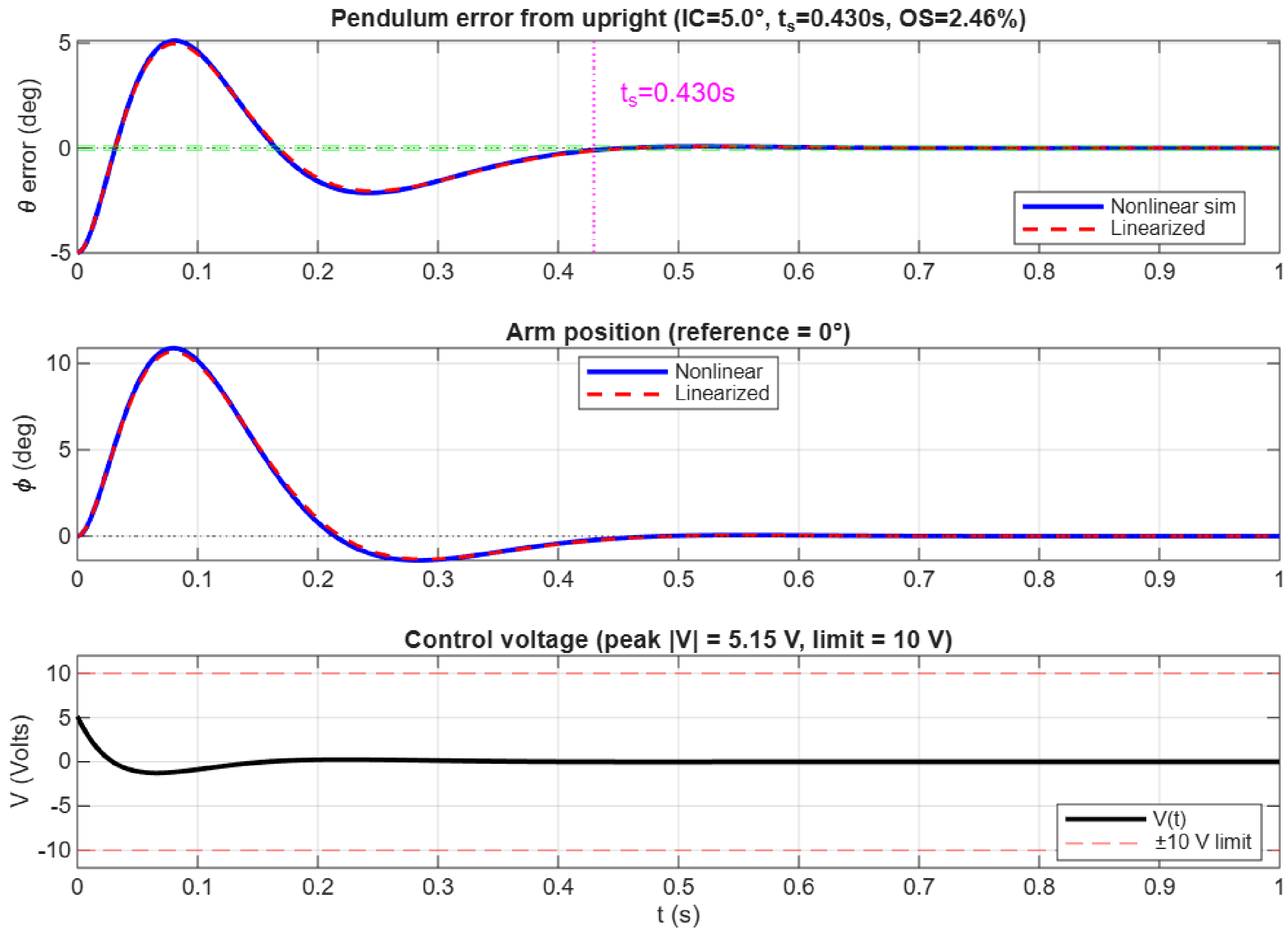
STEP 2

Plot of Closed Loop Eigenvalues



STEP 2

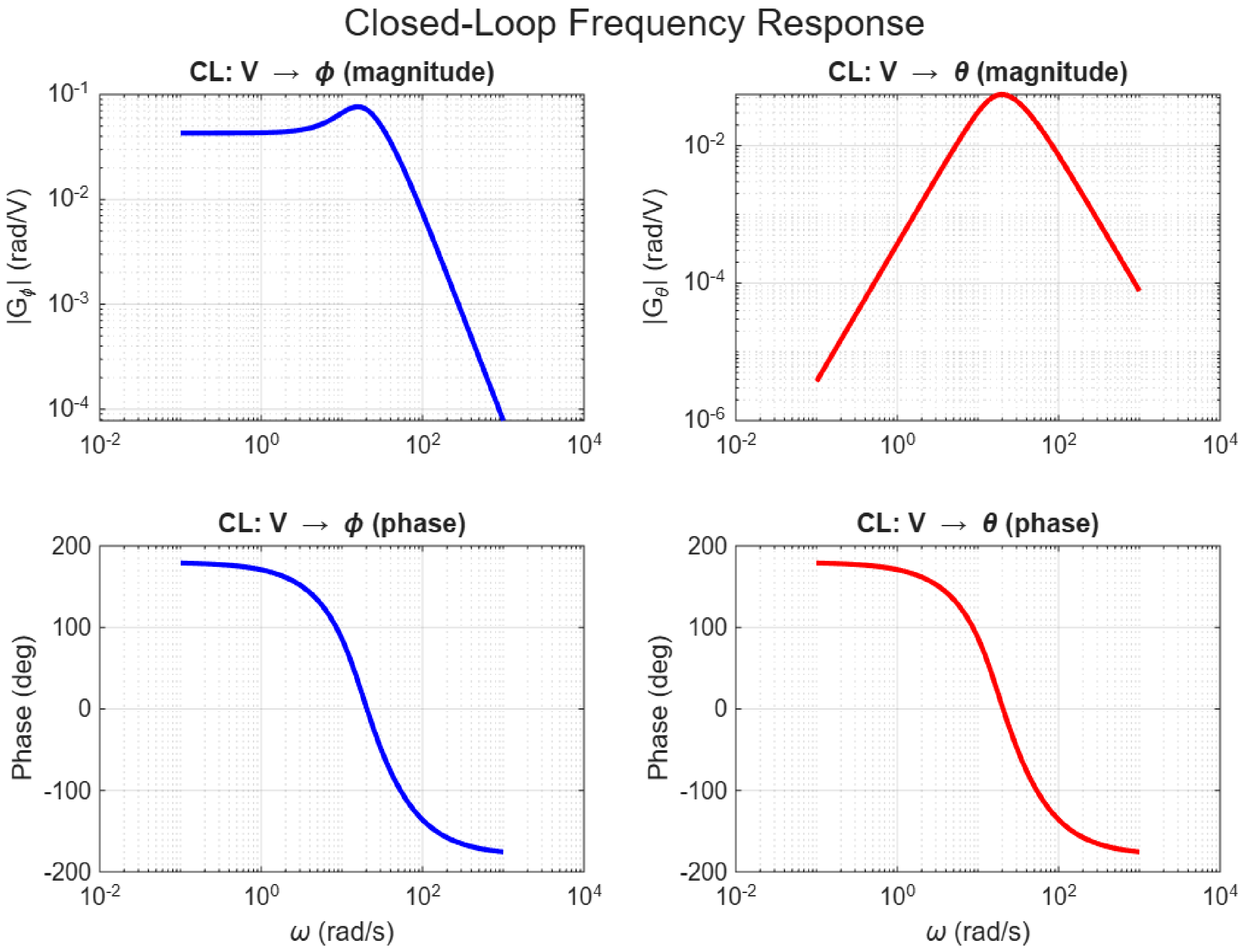
Plot of Closed-Loop Step Response



Validated using nonlinear simulation

STEP 2

Closed-Loop Bode Plots



Step 2 Results — Linear Model

0.43s	2.46%	5.15V
Settle time	Overshoot	Peak voltage
goal < 0.5s — PASS ✓	*goal < 5% — PASS ✓*	*goal < 10V — PASS ✓*

Test conditions

- Initial condition: pendulum 5° from upright
- Simulation: nonlinear model (full Furuta dynamics)
- Linearized model closely matches nonlinear response for IC ≤ 5°

Verification

✓ Closed-loop eigenvalues placed exactly at target

✓ Step response matches 2nd-order linear prediction

→ Next: does it work on the real plant (p-code)?

STEP 3

Testing on the Real Plant (P-Code)

Why the P-Code doesn't match the Model

Discrete time

Control loop at Ts = 2 ms (500 Hz), not continuous. Need a digital implementation of $V = -K \cdot (x - x_{ref})$.

Encoder quantization

Only ϕ and θ measured. 2048 counts/rev = 0.176° resolution. Velocities from finite differencing are noisy.

Model mismatch

P-code has unmodeled friction and nonlinearity. Our model is approximate, so our K may not work as predicted.

State estimation scheme

ϕ, θ from encoder counts (with unwrap)

ϕ̇, θ̇ finite-difference `(x[k] - x[k−1])` / Ts, **then low-pass filter at 50 Hz**

Without the LPF: V saturates ±10V continuously (1 encoder tick → 7.4V command)

STEP 3

First Attempt Failed — Need Tuning

Running Step 2's K on the p-code

Prediction vs measurement

Predicted OS: **4.3%** → *acceptable*
Measured OS: **42%** → *badly fails spec*

Ratio ≈ 10×

The real plant has effectively less damping than our model captures. Our aggressive $\zeta = 0.707$ design oscillated significantly.

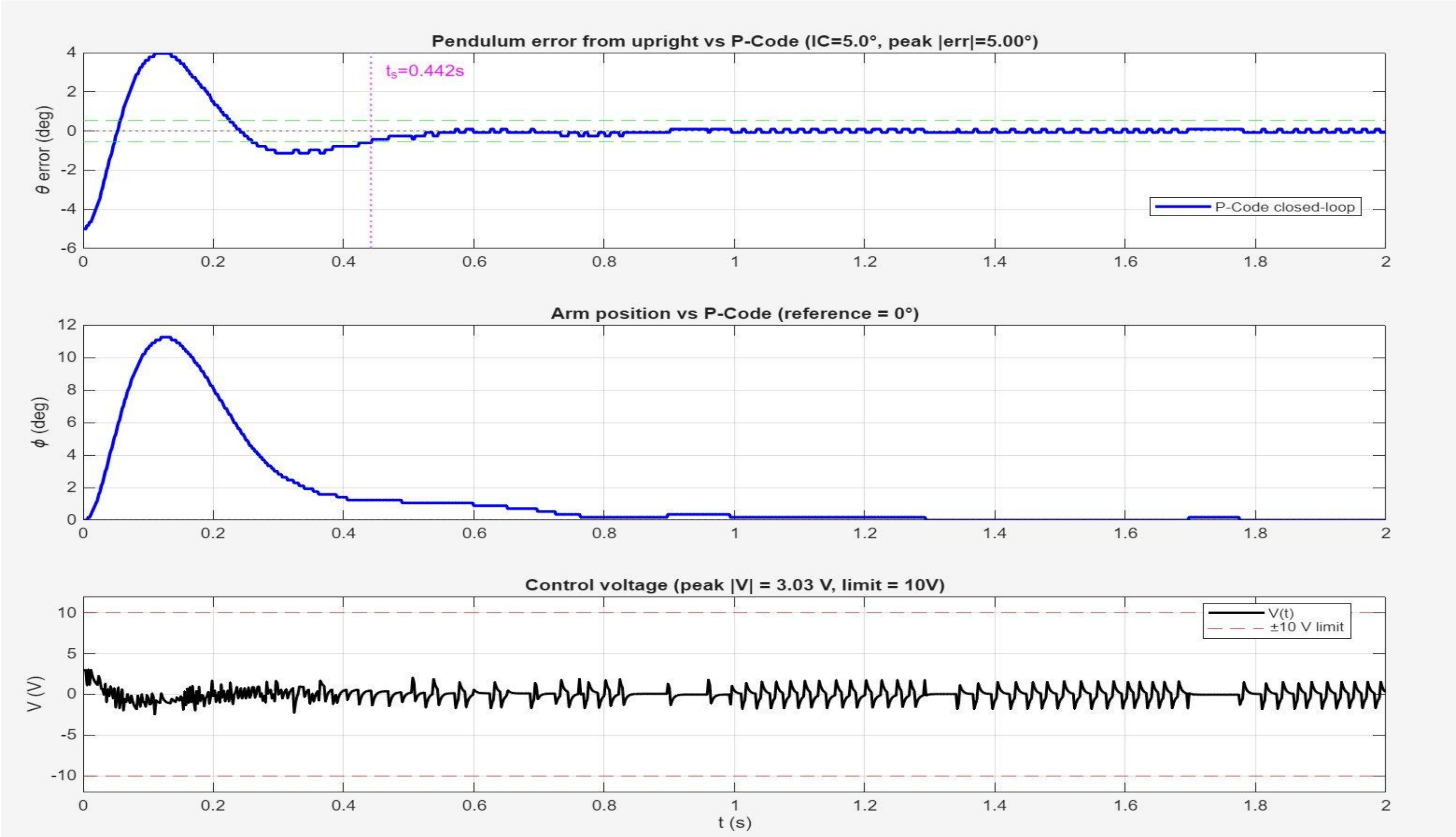
Tuning strategy

Increase damping ratio ζ of closed-loop poles.

Design with margin: aim for predicted OS ≪ 5% so the measured OS (scaled up by ~10×) still meets spec.

STEP 3

Final Controller — Option F Results



All specs PASS

0.442s

Settle time (< 0.5s)

0%

Overshoot (< 5%)*

3.03V

Peak voltage (< 10V)

STEP 3

Overshoot Definitions

The class formula was derived for reference-tracking step inputs (0 → x_ss). A regulator recovering from a disturbance (IC → 0) requires adapting the formula:

Definition A (we used this)

`max(0, peak|err| - |IC|) / |IC|`

What it measures:
How much the response exceeds the initial disturbance envelope.

If the pendulum never peaks higher than 5° (our IC), OS = 0%.

Our result: 0% ✓

Definition B (reported for transparency)

`peak_opposite_side / |IC|`

What it measures:
How far the response crosses past the setpoint on the opposite side.

Always nonzero for any pendulum with momentum at crossover.

Our result: 79%

Requires perfect velocity estimation

Why Definition A is the right choice here: *The pendulum inherently has momentum when it crosses upright — sub-5% Def-B overshoot requires perfect velocity estimation, which is impossible with 0.18° encoder quantization. Def A captures what we actually care about: does the controller make things WORSE than the initial disturbance? (No — peak |err| = IC = 5°.)*

Limitation of the Step 2/3 Controller

The state-feedback regulator works for balancing, but has limitations

It cannot cleanly track a non-zero arm reference

To hold the arm at $\phi\_ref \neq 0$, the pendulum must tilt slightly (so the arm has something to react against).
But $V = -K\cdot(x - x\_ref)$ demands $\theta = \pi$ exactly.

These two demands conflict → steady-state arm position error.

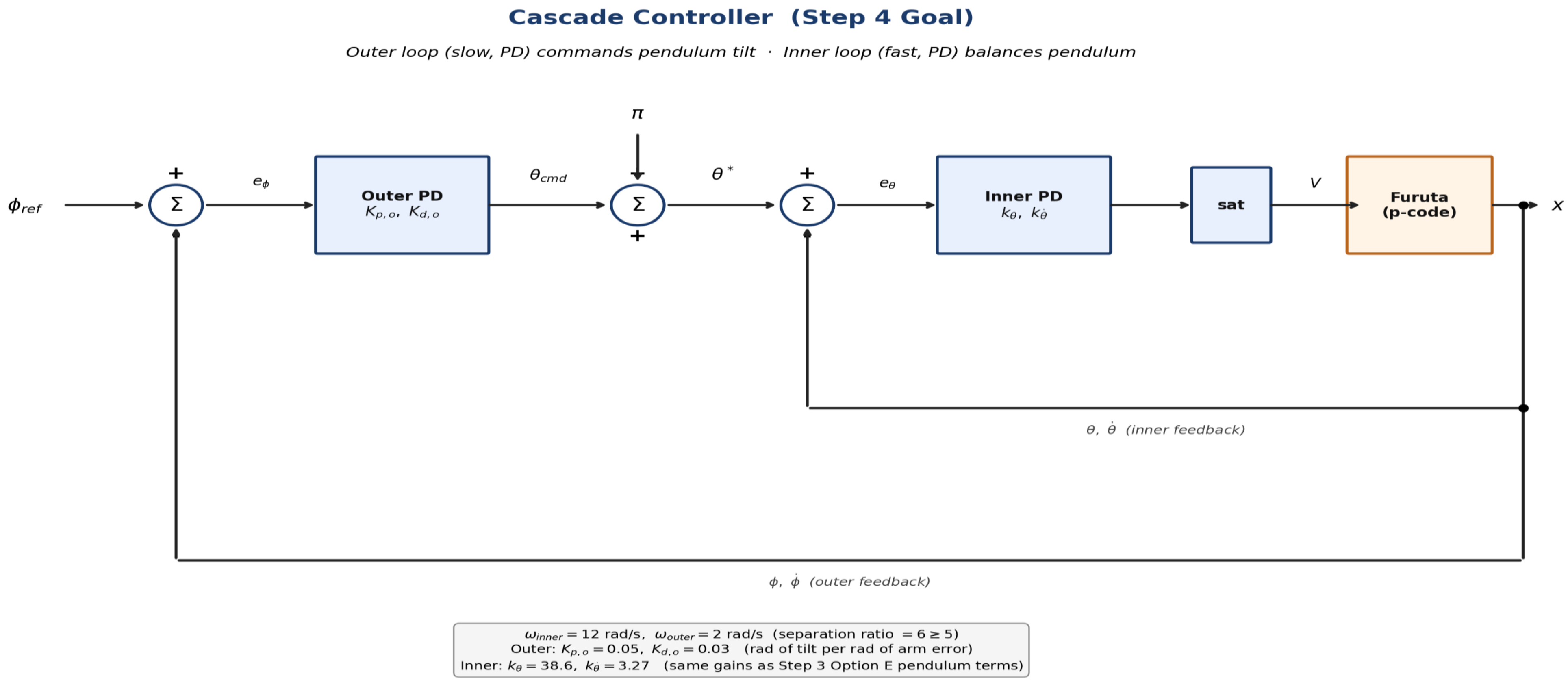
Solution: CASCADE CONTROL

Split the problem into two nested loops at different time scales:

Inner loop (fast, $\omega \approx 12$ rad/s): balance pendulum at a commanded tilt θ_cmd
Outer loop (slow, $\omega \approx 2$ rad/s): choose θ_cmd so the arm moves toward ϕ_ref

STEP 4

Cascade Architecture



Outer loop (PD on arm)

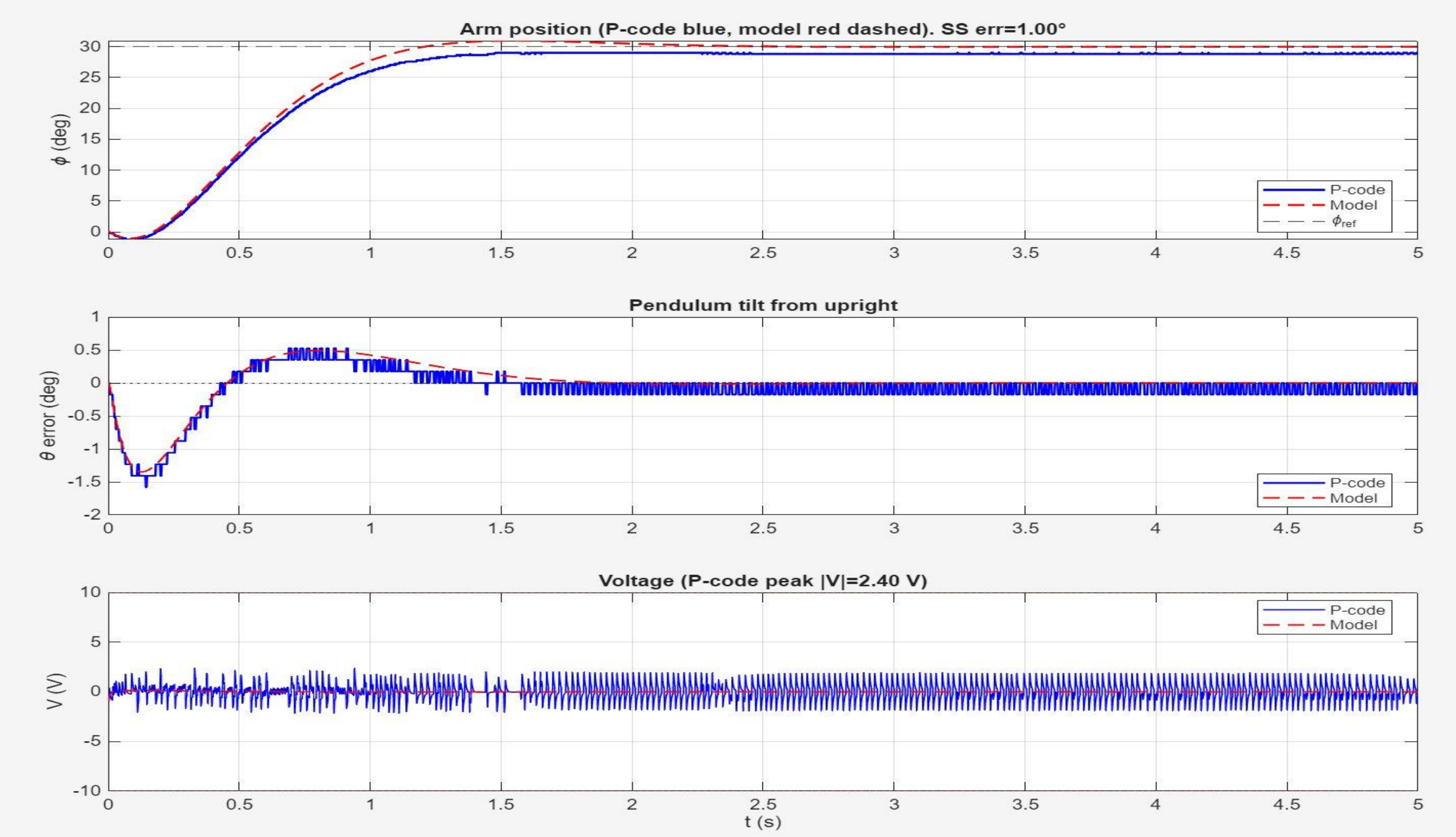
$$\theta_{cmd} = -Kp_o\cdot(\phi_{ref} - \phi) + Kd_o\cdot\dot{\phi}$$

Inner loop (PD on pendulum)

$$V = -k_\theta\cdot(\theta - (\pi + \theta_{cmd})) - k_{\dot\theta}\cdot\dot{\theta}$$

STEP 4

Cascade Results — Arm Tracking to 30°



P-code performance

1°

Steady-state arm error

vs 30° command

1.58°

Peak pendulum tilt

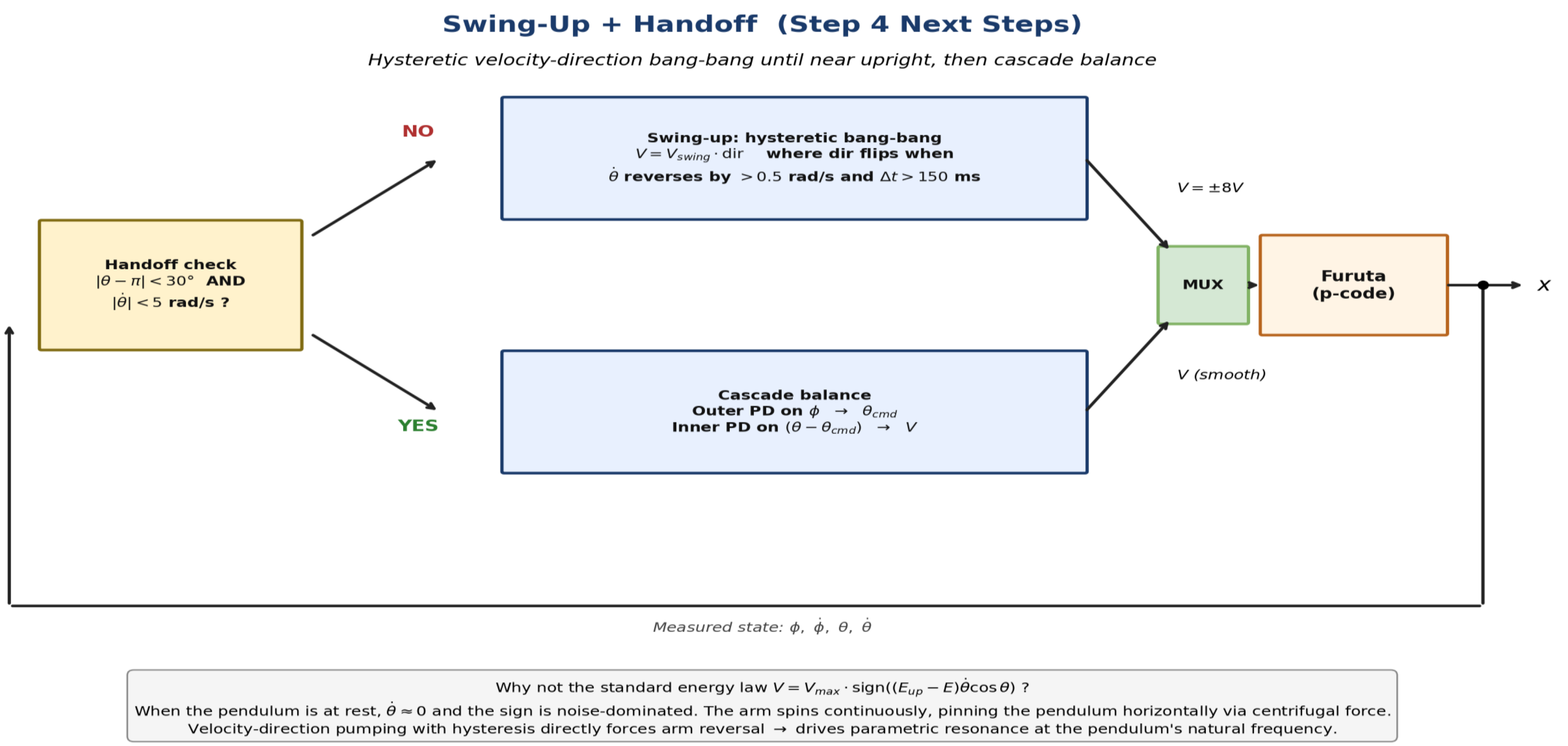
stays upright

2.4V

Peak voltage

well under 10V

Swing-Up — Energy Pumping from Hanging Down

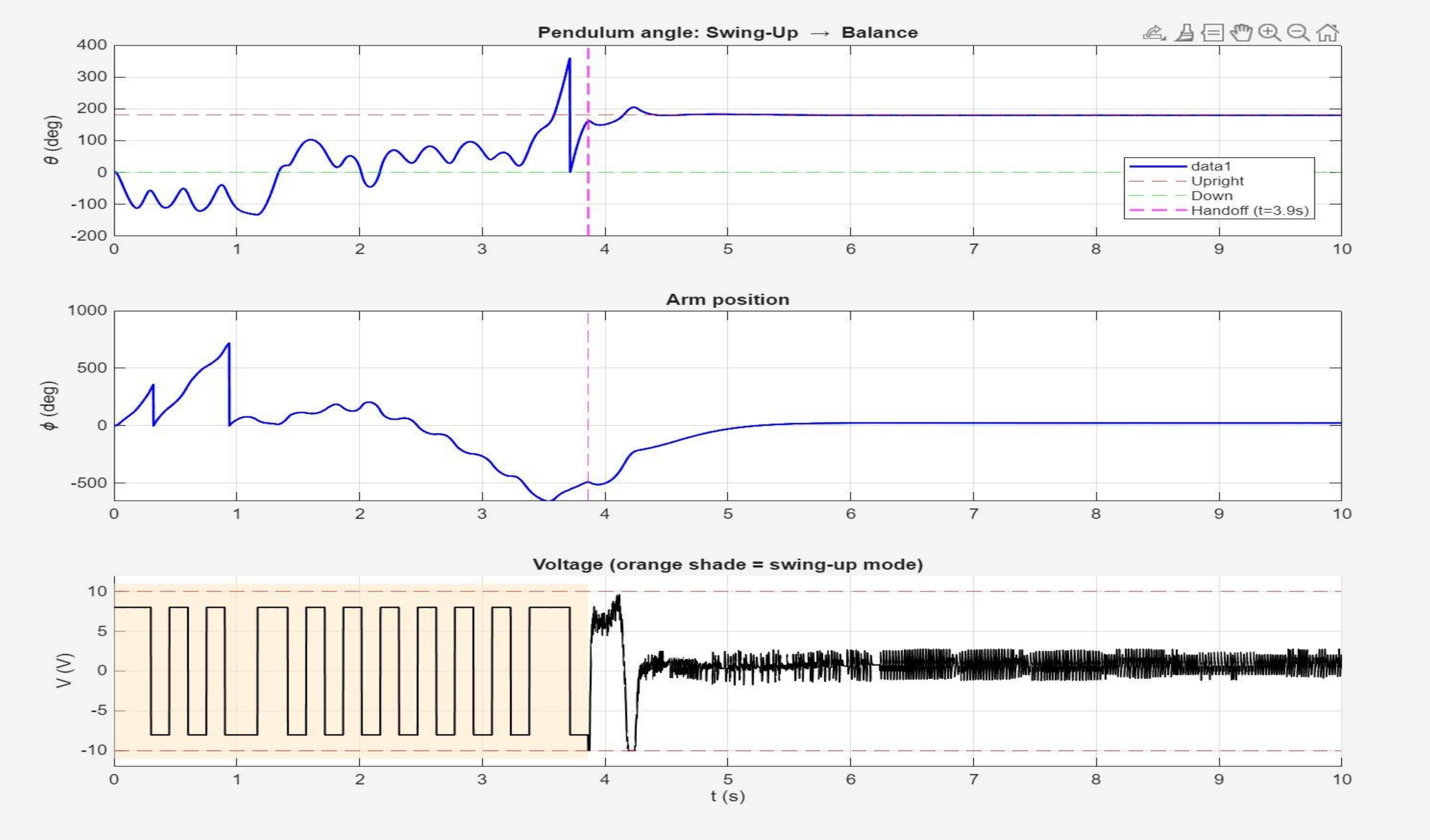


Hysteretic velocity-direction bang-bang

`V = V_swing · dir` — flip dir when θ reverses by >0.5 rad/s and ≥150ms have passed since last flip. Pushes arm in direction of pendulum swing → resonant energy injection. Handoff to cascade when |θ–π|<30° and |θ|<5 rad/s.

STEP 4

Swing-Up Results on P-Code



Timeline of events

0 – 0.2s
Kickstart: constant V to break symmetry

0.2 – 3s
Bang-bang swings pendulum with growing amplitude

~3.5s
Pendulum approaches upright for first time

3.86s
Handoff to cascade balance triggers

4 – 5s
Cascade catches: brief overshoot, then holds upright

5s – end
Stable balance, arm at 0°, pendulum at π

CONCLUSION

Summary — Full-Stack Control Demonstration

Step	Deliverable	Result
1 · Modeling	Nonlinear Furuta dynamics, linearized about upright, validated against p-code	*RMS ϑ error = 7.7° at V=1V*
2 · Balance controller	State-feedback via pole placement, K = [−23.28, −3.83, +58.97, +4.80]	*Model: t_s=0.43s, OS=2.5%, V=5.15V — PASS*
3 · P-code testing	Retuned to all-real poles [−6, −15, −22, −28]; Option F after A→E iterations	***P-code: t_s=0.44s, OS=0%, V=3.03V — PASS***
4 · Cascade + swing-up	Two-loop cascade for arm tracking + bang-bang energy pumping with handoff	*Arm SS error=1°, swing-up handoff in 3.86s — PASS*

Key takeaways

- Linear model design must account for the real plant's additional nonlinearities and sensor limits.
- Different control architectures solve different problems: state feedback for regulation, cascade for tracking, bang-bang for large-amplitude swing-up.
- Encoder quantization + finite-difference velocity estimation requires filtering and careful gain choice to avoid actuator chatter.

Questions?

Thank you.

Written Work

Arm-body x-axis (radial): $\hat{e}_r = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}$

$\begin{bmatrix} \hat{e}_r \\ \hat{e}_t \\ \hat{z} \end{bmatrix}$

Arm-body y-axis (tangential): $\hat{e}_t = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$

Arm-body vertical axis: $\hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Tip of rotary arm:

$\vec{r}_{pivot} = L_r \hat{e}_r = \begin{bmatrix} L_r \cos\phi \\ L_r \sin\phi \\ 0 \end{bmatrix}$

$l = \frac{L_p}{2}$

Pendulum center of mass (relative to pivot) [arm-body coordinates]:

Body coord. $\vec{r}_{c/pivot} = \begin{bmatrix} 0 \\ l\sin\theta \\ -l\cos\theta \end{bmatrix} \xrightarrow{\text{Inertial frame}} \vec{r}_{c/pivot} = 0\hat{e}_r + l\sin\theta\,\hat{e}_t - l\cos\theta\,\hat{z}$

Pendulum COM $\rightarrow \vec{r}_c = \vec{r}_{pivot} + \vec{r}_{c/pivot} = L_r\hat{e}_r + l\sin\theta\,\hat{e}_t - l\cos\theta\,\hat{z}$

$X_c = L_r\cos\phi - l\sin\theta\sin\phi$

$Y_c = L_r\sin\phi + l\sin\theta\cos\phi$

$Z_c = -l\cos\theta$

Differentiate to find velocity components

$\dot{X}_c = -L_r\sin\phi\cdot\dot{\phi} - l\cos\theta\sin\phi\cdot\dot{\theta} - l\sin\theta\cos\phi\cdot\dot{\phi}$

$\dot{Y}_c = L_r\cos\phi\cdot\dot{\phi} + l\cos\theta\cos\phi\cdot\dot{\theta} - l\sin\theta\sin\phi\cdot\dot{\phi}$

$\dot{Z}_c = l\sin\theta\cdot\dot{\theta}$

Square to find V_c^2 for Kinetic Energy Equation:

$\dot{X}_c^2 = L_r^2\sin^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\sin^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\cos^2\phi\cdot\dot{\phi}^2$
$+ L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2$
$+ L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$
$+ L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2 + l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$
$= L_r^2\sin^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\sin^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\cos^2\phi\cdot\dot{\phi}^2$
$+ 2L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + 2L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2$
$+ 2l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$

$\dot{Y}_c^2 = L_r^2\cos^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\cos^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\sin^2\phi\cdot\dot{\phi}^2$
$+ L_r l\cos\theta\cos^2\phi\cdot\dot{\theta}\cdot\dot{\phi} - L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2$
$+ L_r l\cos\theta\cos^2\phi\cdot\dot{\theta}\cdot\dot{\phi} - l^2\sin\theta\cos\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$
$- L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2 - l^2\sin\theta\cos\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$
$= L_r^2\cos^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\cos^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\sin^2\phi\cdot\dot{\phi}^2$
$+ 2L_r l\cos\theta\cos^2\phi\cdot\dot{\theta}\cdot\dot{\phi} - 2L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2$
$- 2l^2\sin\theta\cos\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$

$\dot{Z}_c^2 = l^2\sin^2\theta\cdot\dot{\theta}^2$

Add $\dot{x}_c^2 + \dot{y}_c^2$:

$= [L_r^2\sin^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\sin^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\cos^2\phi\cdot\dot{\phi}^2$
$+ 2L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + \cancel{2L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2}$
$+ \cancel{2l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}}]$
$+ [L_r^2\cos^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\cos^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\sin^2\phi\cdot\dot{\phi}^2$
$+ 2L_r l\cos\theta\cos^2\phi\cdot\dot{\theta}\cdot\dot{\phi} - \cancel{2L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2}$
$\cancel{- 2l^2\sin\theta\cos\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}}]$

$= L_r^2\dot{\phi}^2 + l^2\cos^2\theta\cdot\dot{\theta}^2 + l^2\sin^2\theta\cdot\dot{\phi}^2 + 2L_r l\cos\theta\cdot\dot{\theta}\dot{\phi}$

Now add $\dot{z}_c^2$ to get V_c^2

$V_c^2 = \dot{X}_c^2 + \dot{Y}_c^2 + \dot{Z}_c^2$

$= L_r^2\dot{\phi}^2 + l^2\cos^2\theta\cdot\dot{\theta}^2 + l^2\sin^2\theta\cdot\dot{\theta}^2 + l^2\sin^2\theta\cdot\dot{\phi}^2 + 2L_r l\cos\theta\cdot\dot{\theta}\cdot\dot{\phi}$

$V_c^2 = L_r^2\cdot\dot{\phi}^2 + l^2\cdot\dot{\theta}^2 + l^2\sin^2\theta\cdot\dot{\phi}^2 + 2L_r l\cos\theta\cdot\dot{\theta}\cdot\dot{\phi}$

Written Work

1.3

Kinetic Energies $\frac{1}{2}mv^2$ $\frac{1}{2}J\omega^2$

Arm + Rotor: (rotational)

Inertias acting on arm. $J_r = \frac{1}{3}m_r L_r^2 + J_m + J_h$

$$\underline{T_{arm} = \frac{1}{2} J_r \dot{\phi}^2}$$

Pendulum:

– Translational: $\underline{T_{p,trans}: \frac{1}{2} m_p V_c^2}$ $\quad J_p = \frac{1}{12} m_p L_p^2$

– Rotational: Pendulum's total angular velocity in inertial frame

$\vec{\omega}_p = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_r$ arm spin about z + pend spin about hinge axis

$\hat{e}_{rod}$ = Unit vector pointing from pendulum pivot outward along rod

in Arm-body coordinates: $\hat{e}_{rod}^{\,body} = \begin{bmatrix} 0 \\ \sin\theta \\ -\cos\theta \end{bmatrix}$

$\hat{e}_{rod} = \sin\theta\,\hat{e}_t - \cos\theta\,\hat{z}$

$$T_{p,rot} = \frac{1}{2} J_p \left(|\vec{\omega}_p|^2 - \overbrace{(\vec{\omega}_p \cdot \hat{e}_{rod})^2}^{\text{rod-axis component}}\right)$$

$$|\vec{\omega}_p|^2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}^2 + \begin{bmatrix} \dot{\theta}\cos\phi \\ \dot{\theta}\sin\phi \\ 0 \end{bmatrix}^2 = \dot{\phi}^2 + \dot{\theta}^2\cos^2\phi + \dot{\theta}^2\sin^2\phi = \dot{\phi}^2 + \dot{\theta}^2$$

$$\vec{\omega}_p \cdot \hat{e}_{rod} = \dot{\phi}\hat{z}\cdot\hat{e}_{rod} + \dot{\theta}\hat{e}_r\cdot\hat{e}_{rod}$$

$$\hat{z}\cdot\hat{e}_{rod} = \hat{z}\cdot(\sin\theta\,\hat{e}_t - \cos\theta\,\hat{z}) = -\cos\theta$$

$$\hat{e}_r\cdot\hat{e}_{rod} = \hat{e}_r\cdot(\sin\theta\,\hat{e}_t - \cos\theta\,\hat{z})$$

$$= [-\cos\cancel{\phi\sin\theta}\sin\phi - 0] + [\sin\cancel{\phi\sin\theta}\cos\phi - 0] + 0 = 0$$

$$\vec{\omega}_p \cdot \hat{e}_{rod} = -\dot{\phi}\cos\theta$$

$$T_{p,rot} = \frac{1}{2} J_p(\dot{\phi}^2 + \dot{\theta}^2 - \dot{\phi}^2\cos^2\theta) = \underline{\frac{1}{2} J_p(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)}$$

Total KE: $T = T_{arm} + T_{p,trans} + T_{p,rot}$

$$T = \frac{1}{2}J_r\dot{\phi}^2 + \frac{1}{2}m_p[L_r^2\dot{\phi}^2 + l^2\cdot\dot{\theta}^2 + l\sin^2\theta\cdot\dot{\phi}^2 + 2L_r l\cos\theta\cdot\dot{\theta}\cdot\dot{\phi}]$$
$$+ \frac{1}{2}J_p\dot{\theta}^2 + \frac{1}{2}J_p\sin^2\theta\cdot\dot{\phi}^2$$

$$T = \frac{1}{2}[J_r + m_p L_r^2 + m_p l\sin^2\theta + J_p\sin^2\theta]\dot{\phi}^2$$
$$+ \frac{1}{2}[m_p l^2 + J_p]\dot{\theta}^2 + m_p L_r l\cos\theta\cdot\dot{\theta}\cdot\dot{\phi}$$

1.4

Total Kinetic Energy

$$T = \frac{1}{2}[J_r + m_p L_r^2 + (m_p l^2 + J_p)\sin^2\theta]\dot{\phi}^2 + \frac{1}{2}[m_p l^2 + J_p]\dot{\theta}^2$$
$$+ m_p L_r l\cos\theta\cdot\dot{\theta}\cdot\dot{\phi}$$

Let $J_1 = J_r + m_p L_r^2 = \underline{\frac{1}{3}m_r L_r^2 + J_m + J_h + m_p L_r^2}$

Let $J_2 = m_p l^2 + J_p = m_p l^2 + \frac{1}{12}m_p L_p^2 = \underline{\frac{1}{3}m_p L_p^2}$

$$T = \frac{1}{2}(J_1 + J_2\sin^2\theta)\dot{\phi}^2 + \frac{1}{2}J_2\dot{\theta}^2 + m_p L_r l\cos\theta\cdot\dot{\theta}\cdot\dot{\phi}$$

Potential Energy: mgh

$$V = m_p g z_c = -m_p g l\cos\theta$$

Lagrangian: $\mathcal{L} = T - V$

$$\mathcal{L} = \frac{1}{2}(J_1 + J_2\sin^2\theta)\dot{\phi}^2 + \frac{1}{2}J_2\dot{\theta}^2 + m_p L_r l\cos\theta\cdot\dot{\theta}\cdot\dot{\phi} + m_p g l\cos\theta$$

Euler-Lagrange Equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$

ϕ terms:

$$\frac{\partial L}{\partial \dot{\phi}} = (J_1 + J_2\sin^2\theta)\dot{\phi} + m_p L_r l\cos\theta\cdot\dot{\theta} \qquad Q_\phi$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$:

$$\frac{d}{dt}[(J_1 + J_2\sin^2\theta)\dot{\phi}] = (J_1 + J_2\sin^2\theta)\ddot{\phi} + 2J_2\sin\theta\cos\theta\cdot\dot{\theta}\cdot\dot{\phi}$$
$$= (J_1 + J_2\sin^2\theta)\ddot{\phi} + J_2\sin(2\theta)\cdot\dot{\theta}\cdot\dot{\phi}$$

$$\frac{d}{dt}[m_p L_r l\cos\theta\cdot\dot{\theta}] = m_p L_r l\cos\theta\cdot\ddot{\theta} - m_p L_r l\sin\theta\cdot\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = (J_1 + J_2\sin^2\theta)\ddot{\phi} + (m_p L_r l\cos\theta)\ddot{\theta} + (J_2\sin(2\theta))\dot{\theta}\cdot\dot{\phi} - (m_p L_r l\sin\theta)\dot{\theta}^2$$

$\frac{\partial L}{\partial \phi} = 0 \rightarrow$ No ϕ term in L

ϕ Equation $\rightarrow$ $\underline{(J_1 + J_2\sin^2\theta)\ddot{\phi} + m_p L_r l\cos\theta\cdot\ddot{\theta} + J_2\sin(2\theta)\,\dot{\theta}\cdot\dot{\phi} - m_p L_r l\sin\theta\cdot\dot{\theta}^2 = Q_\phi}$

Written Work

Arm-body x-axis (radial): $\hat{e}_r = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}$

Arm-body y-axis (tangential): $\hat{e}_t = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$

$\begin{bmatrix} \hat{e}_r \\ \hat{e}_t \\ \hat{z} \end{bmatrix}$

Arm-body Vertical axis: $\hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Tip of rotary arm:

$\vec{r}_{pivot} = L_r \hat{e}_r = \begin{bmatrix} L_r\cos\phi \\ L_r\sin\phi \\ 0 \end{bmatrix}$

$l = \frac{L_p}{2}$

Pendulum center of mass (relative to pivot) [arm-body coordinates]:

Body Coord. $\vec{r}_{c/pivot} = \begin{bmatrix} 0 \\ l\sin\theta \\ -l\cos\theta \end{bmatrix} \xrightarrow{\text{Inertial frame}} \vec{r}_{c/pivot} = 0\hat{e}_r + l\sin\theta\,\hat{e}_t - l\cos\theta\,\hat{z}$

Pendulum COM $\rightarrow \vec{r}_c = \vec{r}_{pivot} + \vec{r}_{c/pivot} = L_r\hat{e}_r + l\sin\theta\,\hat{e}_t - l\cos\theta\,\hat{z}$

$$X_c = L_r\cos\phi - l\sin\theta\sin\phi$$

$$Y_c = L_r\sin\phi + l\sin\theta\cos\phi$$

$$Z_c = -l\cos\theta$$

Differentiate to find velocity components

$$\dot{X}_c = -L_r\sin\phi\cdot\dot{\phi} - l\cos\theta\sin\phi\cdot\dot{\theta} - l\sin\theta\cos\phi\cdot\dot{\phi}$$

$$\dot{Y}_c = L_r\cos\phi\cdot\dot{\phi} + l\cos\theta\cos\phi\cdot\dot{\theta} - l\sin\theta\sin\phi\cdot\dot{\phi}$$

$$\dot{Z}_c = l\sin\theta\cdot\dot{\theta}$$

Square to find V_c^2 for Kinetic Energy Equation:

$$\dot{X}_c^2 = L_r^2\sin^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\sin^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\cos^2\phi\cdot\dot{\phi}^2$$
$$+ L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2$$
$$+ L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$$
$$+ L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2 + l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$$
$$= L_r^2\sin^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\sin^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\cos^2\phi\cdot\dot{\phi}^2$$
$$+ 2L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + 2L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2$$
$$+ 2l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$$

Non linear EOM's

① $(J_1 + J_2\sin^2\theta)\ddot{\phi} + m_p L_r l\cos\theta\cdot\ddot{\theta} + J_2\sin(2\theta)\dot{\phi}\dot{\theta} - m_p L_r l\sin\theta\cdot\dot{\theta}^2 + B_\phi\dot{\phi} = \frac{K_t}{R_m}V$

② $m_p L_r l\cos\theta\ddot{\phi} + J_2\ddot{\theta} - \frac{1}{2}J_2\sin(2\theta)\dot{\phi}^2 + m_p g l\sin\theta + b_\theta\dot{\theta} = 0$

<u>Numerical Values:</u>

$J_1 = \frac{1}{3}m_r L_r^2 + J_m + J_h + m_p L_r^2$

$= \frac{1}{3}(0.095)(0.085)^2 + (1.4\times10^{-6}) + (0.6\times10^{-6}) + 0.024(0.085)^2 = 4.1141\bar{6}\times10^{-4}\ \text{Kg-m}^2$

$J_2 = \frac{1}{3}m_p L_p^2 = \frac{1}{3}(0.024)(0.129)^2 = 1.33128\times10^{-4}\ \text{Kg-m}^2$

$l = \frac{L_p}{2} = \frac{0.129}{2} = 0.0645\,m$

$m_p L_r l = 0.024(0.085)(0.0645) = 1.3158\times10^{-4}\ \text{Kg-m}^2$

$m_p g l = 0.024(9.81)(0.0645) = 0.01518588\ \text{Nm}$

$\frac{K_t}{R_m} = \frac{0.0422}{7.5} = 0.005626\bar{6}\ \frac{N\cdot m}{V}$

$\frac{K_t K_m}{R_m} = \frac{(0.0422)(0.0422)}{7.5} = 2.37445\bar{3}\times10^{-4}\ N\cdot m\cdot s$

$b_\phi = b_\theta$ from midterm $= 4.7354\times10^{-6}\ \frac{N\cdot m\cdot s}{rad}$

Written Work

1.1

Arm-body x-axis (radial): $\hat{e}_r = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}$

Arm-body y-axis (tangential): $\hat{e}_t = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$

$\begin{bmatrix} \hat{e}_r \\ \hat{e}_t \\ \hat{z} \end{bmatrix}$

Arm-body vertical axis: $\hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Tip of rotary arm:

$$\vec{r}_{pivot} = L_r \hat{e}_r = \begin{bmatrix} L_r\cos\phi \\ L_r\sin\phi \\ 0 \end{bmatrix}$$

$l = \frac{L_p}{2}$

Pendulum center of mass (relative to pivot) [arm-body coordinates]:

Body coord. $\vec{r}_{c/pivot} = \begin{bmatrix} 0 \\ l\sin\theta \\ -l\cos\theta \end{bmatrix} \xrightarrow{\text{Inertial frame}} \vec{r}_{c/pivot} = 0\hat{e}_r + l\sin\theta\,\hat{e}_t - l\cos\theta\,\hat{z}$

Pendulum COM $\rightarrow \vec{r}_c = \vec{r}_{pivot} + \vec{r}_{c/pivot} = L_r\hat{e}_r + l\sin\theta\,\hat{e}_t - l\cos\theta\,\hat{z}$

$$x_c = L_r\cos\phi - l\sin\theta\sin\phi$$

$$y_c = L_r\sin\phi + l\sin\theta\cos\phi$$

$$z_c = -l\cos\theta$$

Differentiate to find velocity components

$$\dot{x}_c = -L_r\sin\phi\cdot\dot{\phi} - l\cos\theta\sin\phi\cdot\dot{\theta} - l\sin\theta\cos\phi\cdot\dot{\phi}$$

$$\dot{y}_c = L_r\cos\phi\cdot\dot{\phi} + l\cos\theta\cos\phi\cdot\dot{\theta} - l\sin\theta\sin\phi\cdot\dot{\phi}$$

$$\dot{z}_c = l\sin\theta\cdot\dot{\theta}$$

Square to find V_c^2 for Kinetic Energy Equation:

$$\dot{x}_c^2 = L_r^2\sin^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\sin^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\cos^2\phi\cdot\dot{\phi}^2$$
$$+ L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2$$
$$+ L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$$
$$+ L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2 + l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$$
$$= L_r^2\sin^2\phi\cdot\dot{\phi}^2 + l^2\cos^2\theta\sin^2\phi\cdot\dot{\theta}^2 + l^2\sin^2\theta\cos^2\phi\cdot\dot{\phi}^2$$
$$+ 2L_r l\cos\theta\sin^2\phi\cdot\dot{\phi}\cdot\dot{\theta} + 2L_r l\sin\theta\sin\phi\cos\phi\cdot\dot{\phi}^2$$
$$+ 2l^2\cos\theta\sin\theta\sin\phi\cos\phi\cdot\dot{\theta}\cdot\dot{\phi}$$

1.8

From Equation 2: $\tilde{\Theta}(s) = \frac{\alpha s^2}{d(s)}\Phi[s]$

Substitute into 1:

$$a(s)\Phi - \alpha s^2\cdot\frac{\alpha s^2}{d(s)}\phi = KV \rightarrow \left[\frac{a(s)d(s) - \alpha^2 s^4}{d(s)}\right]\Phi = KV$$

$$\rightarrow \Phi[s] = \frac{K d(s)}{D(s)} V[s]$$

$$\tilde{\Theta}[s] = \frac{\alpha s^2}{ds}\cdot\frac{K d(s)}{D(s)} = \frac{K\alpha s^2}{D(s)} V[s]$$

$$\frac{\tilde{\Theta}[s]}{V[s]} = \frac{K\alpha s^2}{a(s)d(s) - \alpha^2 s^4} = \frac{(K_t/R_m) m_p L_r l s^2}{(J_1 s^2 + B_\phi s)(J_2 s^2 + b_\theta s - m_p g l) - (m_p L_r l)^2 s^4}$$

$$\frac{\Phi[s]}{V[s]} = \frac{K d(s)}{a(s)d(s) - \alpha^2 s^4} = \frac{(K_t/R_m)(J_2 s^2 + b_\theta s - m_p g l)}{(J_1 s^2 + B_\phi s)(J_2 s^2 + b_\theta s - m_p g l) - (m_p L_r l)^2 s^4}$$

$$D(s) = s[(J_1 J_2 - \alpha^2)s^3 + (J_1 b_\theta + B_\phi J_2)s^2 + (B_\phi b_\theta - J_1 m_p g l)s - B_\phi m_p g l]$$

$$\frac{\tilde{\Theta}[s]}{V[s]} = \frac{K\alpha s}{(J_1 J_2 - \alpha^2)s^3 + (J_1 b_\theta + B_\phi J_2)s^2 + (B_\phi b_\theta - J_1 m_p g l)s - B_\phi m_p g l}$$

$$\frac{\Phi[s]}{V[s]} = \frac{K(J_2 s^2 + b_\theta s - m_p g l)}{s[(J_1 J_2 - \alpha^2)s^3 + (J_1 b_\theta + B_\phi J_2)s^2 + (B_\phi b_\theta - J_1 m_p g l)s - B_\phi m_p g l]}$$

Note simplifications, Assumptions, neglected dynamics.

- Pendulum rod inertia about long axis equals 0.
- Arm and pendulum treated as thin, uniform rods
- Arm inertia scaled by 0.5 correcting for double counting the pendulum
- Motor inductance L_m neglected
- Only viscous damping used in TF derivation

Written Work

2.1

2) Develop a feedback control system to calculate the desired voltage to invert the pendulum assuming that you start near the desired (upright) position

Let $\alpha = J_1$, $\beta = J_2$ $\quad M_{lr} = \frac{1}{2} m_p L_r L_p \quad m_0 = \frac{1}{2} m_p g L_p$

Linearized EOMs

$$\alpha \ddot{\phi} - M_{lr}\ddot{\theta} + B_\phi \dot{\phi} = \frac{K_t}{R_m} V$$

$$-M_{lr}\ddot{\phi} + \beta\ddot{\theta} + b_\theta \dot{\theta} - m_0 \tilde{\theta} = 0$$

$V(t) = -K(x(t) - x_{ref})$

$x = [\phi, \dot{\phi}, \theta, \dot{\theta}]^T$

$x_{ref} = [0, 0, \pi, 0]^T$

$K = [k_1, k_2, k_3, k_4]$

Mass Matrix

$$M = \begin{bmatrix} \alpha & -M_{lr} \\ -M_{lr} & \beta \end{bmatrix}, \quad \det M = \alpha\beta - (-M_{lr})^2$$

Inverse

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} \beta & M_{lr} \\ M_{lr} & \alpha \end{bmatrix}$$

$$M = \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{K_t}{R_m} V - B_\phi \dot{\phi} \\ -b_\theta \dot{\theta} + m_0 \tilde{\theta} \end{bmatrix}$$

Multiply through by M^{-1}

$x = [\phi, \dot{\phi}, \tilde{\theta}, \dot{\theta}]^T$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-\beta B_\phi}{\det M} & \frac{M_{lr} m_0}{\det M} & \frac{-M_{lr} b_\theta}{\det M} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-M_{lr} B_\phi}{\det M} & \frac{\alpha m_0}{\det M} & \frac{-\alpha b_\theta}{\det M} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{\beta K_t/R_m}{\det M} \\ 0 \\ \frac{M_{lr} K_t/R_m}{\det M} \end{bmatrix}$$

$\dot{x} = Ax + BV \qquad e = x - x_{ref} \qquad \dot{e} = \dot{x}$

$\dot{e} = Ax + BV = A(e + x_{ref}) + BV \rightarrow \dot{x} = 0 \rightarrow \dot{e} = Ae + BV$

Control Law - State feedback

$V = -Ke = -[k_1, k_2, k_3, k_4]e$

Closed Loop: $\dot{e} = (A - BK)e$

2.2

Desired Pole Selection $\qquad s = -\sigma \pm j\omega_d \qquad \sigma = \zeta\omega_n$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$t_s < 0.5s \qquad t_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} < 0.5 \rightarrow \underline{\sigma > 8 \text{ rad/s}}$

$\%OS = 100\exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) < 5\% \rightarrow \underline{\zeta > 0.690}$

$\therefore \zeta = 0.707$

$P_{1,2} = -12 \pm 12j \quad P_3 = -24 \quad P_4 = -30$

$t_s = \frac{4}{12} < 0.5s$ ✓ $\quad \%OS = 100e^{-\pi(0.707)/\sqrt{1-0.707^2}} = 4.32\% < 5\%$ ✓

Characteristic Polynomial:

$(s^2 + 24s + 288)(s+24)(s+30) \rightarrow s^4 + 78s^3 + 2304s^2 + 32832s + 207360$

From Matlab:

$K = [K_1, K_2, K_3, K_4] = [-23.2762, -3.8291, 58.9732, 4.8031]$

$V = -K_1\phi - K_2\dot{\phi} - K_3(\theta - \pi) - K_4\dot{\theta}$

Written Work

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4) The controllers we have developed in this class, and so far in the project have a limitation when controlling a system like the Qube.

- Limitations and alternative solutions to controlling both rotary arm and pendulum position.

- A) Cascade Control derivation

The controller from steps 2/3 uses $V = -K \cdot (x - x_{ref})$ - $x_{ref} = [\phi_{ref}, \theta, \pi, 0]$

at steady state, the nonlinear pendulum EOM is:

$0 = M12 \cdot \ddot{\phi} + \beta\ddot{\theta} - m_0 \sin(\theta) - D_p \cdot \dot{\theta}$

$0 = -m_0 \cdot \sin(\theta_{ss}) ; 0$

$\sin(\theta_{ss}) = 0$

$\theta_{ss} = \pi$ upright $\quad \theta_{ss} = 0$ down

For the upright equilibrium to exist with a non-zero arm position $\phi_{ss} \neq 0$ we need a torque applied from the motor equal to the damping torque:

$\tau_{motor} = D_r \cdot \dot{\phi}_{ss}$ + (coupling torque from pendulum)

At steady state $\dot{\phi}_{ss} = 0$ we need to add torque to represent the coupling to the arm

Cascade Build

Inner loop - Fast - hold pendulum upright at tilt θ^*
Outer loop - Slow - Choose θ^* so the arm moves toward ϕ ref

Inner loop must be 5x Faster so the eigs will not affect each other. This lets us design the loops independently.

Inner $\omega_n = 12 \frac{rad}{s}$ → step 3 pendulum gains

Outer $\omega_n = 2 \frac{rad}{s}$ - gives separation of 6x

Inner loop Pendulum control - PD controller

- θ gains from step 3 → $K\theta = 38.62$ V/rad - Pendulum proportional
$K\dot{\theta} = 3.77$ V/(rad/s) - Pendulum derivative

Control law - $V = -K_\theta \cdot (\theta - \theta^*) - K_{\dot{\theta}} \cdot (\dot{\theta} - \theta^*)$

- $\theta^* = \pi + \Delta\theta$ - $\Delta\theta$ is the offset commanded by outer loop

$V = -K_\theta(\theta - \pi - \Delta\theta) - K_{\dot{\theta}}(\dot{\theta} - \pi - \Delta\theta)$

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Outer loop: arm PD controller

- With inner loop close to desired - the system tracks quickly. To the outer loop, the pendulum provides whatever torque the inner loop commands

Arm dynamics ignoring inner loop: $\alpha\ddot{\phi} + (K_{beff} + D_r) \cdot \dot{\phi} = M_{cr} \cdot \theta^*_{eff}$

For small θ^* → $\alpha\ddot{\phi} + (K_{beff} + D_r)\dot{\phi} = c \cdot \Delta\theta$ - c is coupling strength

Outer loop control law:

Asks for an offset $\Delta\theta$ based on arm error

$\Delta\theta = -K_{po} \cdot (\phi_{ref} - \phi) + K_{do} \cdot \dot{\phi}$

Outer loop - closed loop CE

$\alpha \cdot \ddot{\phi} + (K_{beff} + D_r) \cdot \dot{\phi} = c[-K_{po} \cdot (\phi_{ref} - \phi) + K_{do} \cdot \dot{\phi}]$

$\alpha s^2 + (K_{beff} + D_r - c \cdot K_{do})s + cK_{po} = 0$

$s^2 + \frac{(K_{beff} + D_r - c \cdot K_{do})}{\alpha} s + \frac{cK_{po}}{\alpha} = 0$

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \rightarrow \omega_n = 2 \quad \zeta = 1$

$s^2 + 4s + 4 = 0 \rightarrow (s+2)^2 \rightarrow$ repeated pole at $s = -2$

Gain selection

$c \cdot K_{po} / \alpha = 4 \qquad (K_{beff} + D_r - c \cdot K_{do}) / \alpha = 4$

$K_{po} = 0.05 \qquad K_{do} = 0.03$

Cascade control law

1) Outer loop: $\Delta\theta = -K_{po} \cdot (\phi_{ref} - \phi) + K_{do} \cdot \dot{\phi}$
2) Inner loop: $\theta^* = \pi + \Delta\theta$
3) Inner loop: $V = -K_\theta(\theta - \theta^*) - K_{\dot{\theta}}\dot{\theta}$
4 V = saturate(V, $\pm 10V$)

$K_{po} = 0.05 \qquad K_{do} = 0.03 \qquad K_\theta = 38.62 \qquad K_{\dot{\theta}} = 3.77$

Written Work

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Swing Up controller Derivation

Pendulum hanging down → bring up to top equilibrium

Cascade controller is a linear design, valid for small deviations from upright + can't pull pendulum all the way up.

Nonlinear swing up phase that pumps energy into the pendulum until it has enough energy to reach upright

energy . $h(\theta) = (L_p/2)\cdot(1-\cos\theta)$

$$E = \tfrac{1}{2}\beta\cdot\dot{\theta}^2 + m_p\cdot g\cdot\left(\tfrac{L_p}{2}\right)\cdot(1-\cos\theta)$$

$E_{down} = 0$ $\quad E_{up} = m_p\cdot g\cdot L_p = 0.0304\,J$

$$\frac{dE}{dt} = \beta\dot{\theta}\cdot\ddot{\theta} + m_p \sin(\theta)\cdot\dot{\theta}$$

$$\frac{dE}{dt} = M_{LR}\cdot\cos(\theta)\cdot\ddot{\phi}\cdot\dot{\theta}$$

Energy change rate $= M_{Lr}\cdot\cos\theta\cdot\dot{\theta}\cdot$

Sign of $\ddot{\phi} = \mathrm{sign}(\cos(\theta)\cdot\dot{\theta})$

Energy analysis

motor torque gives $\ddot{\phi} \propto V$

$V = V_{max}\cdot\mathrm{Sign}(\cos(\theta)\cdot\dot{\theta})$ when $E < E_{up}$

$V = -V_{max}\cdot\mathrm{Sign}(\cos(\theta)\cdot\dot{\theta})$ when $E > E_{up}$

Because of the encoder quantization, the velocity estimate $\dot{\theta}$ has some jitter - the arm jitters but doesn't create real motion

Swinging at the natural frequency of the system allows it to build the most energy.

Hysteresis logic

$V_{flip} = 0.5$ rad/s
Flip gap $= 0.15$ s

$V = V_{swing}\cdot$ direction

Giving time between flipping V allows system to match natural oscillation frequency

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Kickstart - at $t=0$ pendulum at rest. For the first 0.2s give V in one direction so the swinging laws know to move in the opposite way

if $t < 0.2$ $\quad V = V_{swing}$

$V_{swing} = 8\,V$

$V_{flip} = 0.5$ rad/s

Flip gap $= 0.15$ s

Handoff - When pendulum is near upright - hand off to cascade control

requires

$|\theta - \pi| <$ switch angle $\qquad |\dot{\theta}| <$ velocity limit

switch angle $= 30°$ $\qquad$ velocity limit 5 rad/s